Course introduction

1. In calculus, you learned about 2nd-order Taylor series so that

$$f(x) = f(x_0) + f^{(1)}(x_0)(x - x_0) + {}^1/_2\, f^{(2)}(x)(x - x_0)^2 + {}^1/_6\, f^{(3)}(\xi)(x - x_0)^3.$$

Consequently, we have that, if we apply this to the functions $\sin(x)$ and $\cos(x)$ at $x_0 = 0$, we have

$$\sin(x) \approx x \text{ and } \cos(x) \approx 1 - \frac{x^2}{2}.$$

What is the absolute error, the relative error and percent relative error of these approximations when $x = 1.0$, 0.1 and 0.01?

Answer:

> 0.1585, 0.1884 and 18.84%
> 0.0001666, 0.001669 and 0.1669%,
> 0.0000001667, 0.00001667 and 0.001667%
>
> 0.04030, 0.07459, 7.459%
> 0.000004165, 0.000004186, 0.0004186%
> 0.0000000004167, 0.0000000004167, 0.00000004167%
>
> The approximation for the cosine function is better because it is based on two terms of the Taylor series, while the sine approximation only uses one term of the Taylor series.

2. Is the 2nd-order Taylor series approximation more accurate for $\sin(x)$ or $\cos(x)$?

Answer: It appears to be more accurate for $\sin(x)$.

3. The following infinite series $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$, and thus $\sum_{k=1}^{n} \frac{1}{k^2}$ must be approximations of $\frac{\pi^2}{6}$. What are the absolute, relative and percent relative errors of these approximations when $n = 2$, 3 and 4?

Answer:

> 0.3949, 0.2401 and 24.01%
> 0.2838, 0.1725 and 17.25%
> 0.2213, 0.1345 and 13.45%

4. How large must n be in Question 3 so that the percent relative error is less than 0.1%? You will need a computer to find n.

Answer: 608

5. Would you suggest that the given algorithm in Question 3 is an accurate means of approximating π?

Answer: Yes, because as n becomes large, the error does go to zero.

6. Would you describe the algorithm in Question 3 as a precise means of approximating π?

Answer: Precision can only be used to differentiate between two algorithms, but this algorithm does not appear to be very precise. When $n = 608$, the approximation of π is 3.1400. A more precise algorithm for approximating π is the Gauss-Legendre algorithm, and an even more precise algorithm (for the same number of steps) is the Ramanujan-Sato series.

7. If $n = 1\,000\,000$, the approximation of π in Question 3 is 3.141591698660467. Approximately how many significant digits does this approximation have?

Answer: The correct answer is **3.14159**2653589793, so it seems to have approximately six significant digits.

8. Which has more significant digits of precision in approximating 10? 10.001 or 9.999999?

Answer: While none of the actual digits are correct, 9.999999 has more significant digits. After, all, if you were to round 9.99999 to six significant digits, you would get 10.0000, which has six matching digits (note "matching digits", not "significant digits") while 10.001 has only four matching digits.

9. Round each of the following to three significant digits:

> 1574999, 1575000, 1585000
> 83.55000, 83.65000, 83.54999, 83.4500, 83.45000001
> 0.01925000, 0.01934999, 0.01925001, 0.01935000

Answer:

> 1.57×10^6, 1.58×10^6, 1.58×10^6
> 8.36×10^1, 8.36×10^1, 8.35×10^1, 8.34×10^1, 8.35×10^1
> 1.92×10^{-2}, 1.93×10^{-2}, 1.93×10^{-2}, 1.94×10^{-2}

10. Why is using scientific notation so relevant when showing a rounded number?

Answer: Consider rounding 15838135 to three digits. The result would be 15800000; however, looking at the second number, you don't know if this number was a number rounded to three digits or eight, after all, 15799999.5 would also round to 15800000; however, 1.58×10^7 makes it clear there are three significant digits, whereas eight significant digits would be represented as 1.5800000×10^7.

Acknowledgement: Chenao Yuan found a typo in Question 1.